

The Four Dimensional Green-Schwarz Mechanism and Anomaly Cancellation Conditions

Francisco Gonzalez-Rey ¹

Institute for Theoretical Physics
State University of New York
Stony Brook, N. Y. 11794-3840

Abstract

We consider a theory with gauge group $G \times U(1)_A$ containing: i) an abelian factor for which the chiral matter content of the theory is anomalous $\sum_f q_A^f \neq 0 \neq \sum_f (q_A^f)^3$; ii) a nonanomalous factor G . In these models, the calculation of consistent gauge anomalies usually found in the literature as a solution to the Zumino-Stora descent equations is reconsidered. Another solution of the descent equations that differs on the terms involving mixed gauge anomalies is presented on this paper. The origin of their difference is analysed, and using Fujikawa's formalism the second result is argued to be the divergence of the usual chiral current. Invoking topological arguments the physical equivalence of both solutions is explained, but only the second one can be technically called the consistent anomaly of a classically invariant theory. The first one corresponds to the addition of noninvariant local counterterms to the action. A consistency check of their physical equivalence is performed by implementing the four dimensional string inspired Green-Schwarz mechanism for both expressions. This is achieved adding slightly different anomaly cancelling terms to the original action, whose difference is precisely the local counterterms mentioned before. The complete anomaly free action is therefore uniquely defined, and the resulting constraints on the spectrum of fermion charges are the same. The Lorentz invariance of the fermion measure in four dimensions forces the Lorentz variation of the Green-Schwarz terms to cancel by itself, producing an additional constraint usually overlooked in the literature. This often happens when a dual description of the theory is used without including all local counterterms.

¹email: glezrey@insti.physics.sunysb.edu

1 Introduction

The issue of anomaly cancellation is one of crucial importance in field theory, since the presence of nonzero anomalies breaks the gauge invariance of the quantum theory. This fact is best understood when the anomaly is seen as a Jacobian factor, arising from a gauge transformation of the path integral measure factor corresponding to the chiral fermions of an otherwise classically (Lagrangian) gauge invariant theory. The powerful path integral formalism requires regularization of the anomaly term, since it is in principle an ill defined expression [1]. The issues of the regularization independence of the method and related problems have been studied by various authors. We will not attempt to discuss these problems, but merely derive the expression for the anomaly that can also be obtained from diagrammatic calculation.

A different approach, more algebraic, was developed in reference [2] using the properties of the gauge group. Regarding the anomaly as the result of a gauge variation in the effective action, its functional form must be consistent with the fact that the commutator of two gauge transformations can be reproduced by a gauge transformation involving the structure constants of the group. In this approach, local functionals in gauge fields and their derivatives obeying this condition are considered to be acceptable expressions of the anomaly density. A method of obtaining solutions for the consistency condition was developed in reference [4], but this solution is not always unique. In the case we are considering it is only defined up to the gauge variation of local functionals of gauge fields and their derivatives. Such arbitrariness is usually justified by arguing that the effective action has certain degree of indetermination since it can only be perturbatively defined up to such local functionals. The fact is that different anomaly expressions correspond to different classical actions, obtained by adding these functionals to a classically invariant action to make it noninvariant. Presumably we will obtain these terms as higher order string loop effects, while the gauge invariant action corresponds to the lowest order low energy limit of the string theory. The corresponding redefined effective field theories are considered to be physically equivalent, and each author uses the anomaly expression of his choice, very often forgetting that the action is no longer the original invariant one.

As long as the theory is considered an effective one that point of view is acceptable. However it is very common to find computation of the anomalies of a classically invariant theory using the solutions of the consistency conditions. In this case the consistent anomaly can only come from the transformation of the path integral measure, and it is unambiguously determined. This is actually the only piece in the gauge variation of any effective action that cannot be removed by adding local counterterms to it, and this is often thought to imply that such expression of the anomaly is the only physically meaningful one. We will determine in this paper which solution of the descent equations corresponds to the invariant Lagrangian. The fact that this invariant classical action can be understood to be physically equivalent to the ones corresponding to other solutions of the descent equations, will be analysed from a topological point of view. We will show that there is a common topological number associated to the fermion measure anomaly.

A necessary condition of the physical equivalence of these theories is that implementation of the Green-Schwarz mechanism must lead to the same conditions in the spectrum of chiral fermion charges. The mechanism mentioned involves completing the original action

with gauge noninvariant terms that cancel its gauge variation. The final complete action is therefore always the same, and the cancellation conditions should not change.

In the Standard Model anomalies are not problematic. The sum of the $SU(3) \times SU(2) \times U(1)_Y$ anomaly coefficients over all the chiral matter content of the theory adds up to zero, and it is believed that any extension including additional irreducible chiral representations of the gauge group must have charges that balance the anomaly coefficients to zero.

However, in recent years the possibility of having extra abelian gauge symmetries, obtained along with the SM gauge group after compactification of a 10-dimensional superstring model with a large gauge group $SO(32)$ or $E_8 \times E_8$ and additional background Wilson lines on a 6-dimensional manifold, has been considered in different models. Quite generally [18] the low energy field theory limit of the compactified superstring has a chiral matter content that makes one of these abelian symmetries anomalous. It is also a common feature in 4-dimensional free-fermionic string constructions with a factorized gauge group containing abelian factors [19]. Since the underlying string theory is anomaly free it is believed that a consistent effective field theory has to be also anomaly free.

It is important therefore to have an accurate computation of the possible anomalies of a theory to study their cancellation. We will review now the field theory calculation of the anomaly expression associated with the gauge group $G \times U(1)_A$.

In section two we give our working definition of the fermion measure anomaly. In section three different solutions of the descent equations are presented, for G abelian and nonabelian. In section four, one of them is determined to be the fermion measure anomaly using the chiral version of Fujikawa's regularization. In section five we give a topological interpretation of the physical equivalence of the theories corresponding to different solutions. In section six we study the Green-Schwarz anomaly cancellation mechanism, putting some emphasis in the rigorous derivation of the constraints imposed on the spectrum of anomalous chiral fermion charges. In section seven we transform the theory into a dual version common in the literature, to explain why one of the cancellation conditions is usually overlooked. We also examine some consequences of the presence of an anomalous $U(1)_A$ that seem to support the cancellation condition mentioned above.

2 Definition of the anomaly

We must define the anomaly in both the Fujikawa and the Wess-Zumino approach, to understand the difference between them.

If we perform a local gauge transformation with parameter $\alpha(x) = \alpha^a(x)T^a$ on the fields of a Yang-Mills theory coupled to chiral fermions in the fundamental representation of $G \times U(1)_A$ (so that $T^a = \lambda^a$ antihermitian for a G transformation and $T^a = q_A$ for a $U(1)_A$ transformation), using the fact that these fermions are chiral we can rewrite their gauge transformation as a chiral transformation with the charge containing the chirality sign

$$\psi'_L = \exp(-\alpha(x))\psi_L = \exp(-\alpha(x)\gamma_5\gamma_5)\psi_L = \exp(-\alpha(x)\gamma_5)\psi_L \quad (1)$$

$$\psi'_R = \exp(-\alpha(x))\psi_R = \exp(-\alpha(x)\gamma_5\gamma_5)\psi_R = \exp(+\alpha(x)\gamma_5)\psi_R \quad (2)$$

so in general we will write the gauge transformation acting on chiral fermions as

$$\psi'_L = \exp(-n_L \alpha(x) \gamma_5) \psi; \bar{\psi}'_L = \bar{\psi}_L \exp(-n_L \alpha(x) \gamma_5) \quad (3)$$

Following the analysis of Fujikawa [1], we can compute the infinitesimal gauge variation of the path integral of the chiral Euclidean theory and we will find an extra term coming from the Jacobian of the transformation in the measure

$$\begin{aligned} Z' &= \int \prod_x DA'_\mu(x) D\bar{\psi}'(x) D\psi'(x) e^{-S(A'_\mu, \bar{\psi}', \psi')} \\ &= \int \prod_x DA_\mu(x) D\bar{\psi}(x) D\psi(x) \left(1 + \int d^4x \text{tr} \alpha(x) A(x) - \delta S(A_\mu, \bar{\psi}, \psi)\right) e^{-S(A_\mu, \bar{\psi}, \psi)} \end{aligned} \quad (4)$$

If $\delta S = 0$, a nonvanishing anomaly $A(x) \neq 0$ makes the gauge invariance break at the quantum level. In this case, since the anomalous variation corresponds to the fermion measure only, we can restrict our study to the gauge variation of the effective action

$$\int D\bar{\psi} D\psi \exp \left(- \int d^4x i\bar{\psi} \not{D}(A) P_L \psi \right) = \exp \Gamma(A); \quad \delta \Gamma(A) = \int d^4x \alpha^a D_\mu \frac{\partial \Gamma(A)}{\partial A_\mu^a} \quad (5)$$

Since we can rewrite

$$\frac{\partial \Gamma(A)}{\partial A_\mu^a} = \langle -i\bar{\psi} \gamma^\mu \lambda^a P_L \psi \rangle = J_L^{\mu a} \quad (6)$$

in the Fujikawa approach we will identify the gauge anomaly density with the covariant divergence of the usual chiral gauge current $(D_\mu J_L^\mu)^a$. For an abelian anomaly we will identify it with the divergence $\partial_\mu J_L^\mu$.

If we add local terms in gauge fields to the action, the gauge variation of the new action will correspond to the divergence of a different current $\partial \Gamma(A)/\partial A_\mu^a$, so we should not confuse it with the previous definition.

3 Solutions of the descent equations

We will compute now the solutions of the descent equations for our factor gauge group.

In reference [4] it was shown that for a simple nonabelian gauge group the chiral anomaly $\alpha(x)A(x)$ obtained by Bardeen [5] through diagrammatic calculation and verifying the Wess-Zumino consistency conditions

$$(\delta_{\theta^1} \delta_{\theta^2} - \delta_{\theta^2} \delta_{\theta^1}) Z = \delta_{[\theta^1, \theta^2]} Z \quad (7)$$

can be reproduced by the solution of the descent equation

$$\delta \omega_{2n+1} = d\omega_{2n}^1 \quad (8)$$

where ω_{2n+1} is the generalized Chern-Simons form defined by the conveniently normalized $2n+2$ dimensional Chern character $1/192\pi^2 \text{tr} F^{n+1} = d\omega_{2n+1}$. For a nonabelian simple gauge group in $2n = 4$ dimensions this gives the well known result

$$tr\alpha(x)D_\mu J_L^\mu = \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}tr\alpha(x)\partial_\mu(A_\nu\partial_\rho A_\sigma + \frac{1}{2}A_\nu A_\rho A_\sigma) \quad (9)$$

We can try to use the descent equations for our factor gauge group with field strength $F_{\mu\nu} = [D_\mu, D_\nu] = q_A^f B_{\mu\nu} + \lambda_{Ga} F_{\mu\nu}^a$, and see if we reproduce the fermion measure anomaly. The Zumino-Stora method would start with the following Bose symmetric Chern character in 6 dimensions

$$\begin{aligned} & tr\{(q_A^f B_{\mu\nu} + F_{\mu\nu})(q_A^f B_{\rho\sigma} + F_{\rho\sigma})(q_A^f B_{\alpha\beta} + F_{\alpha\beta})\}\epsilon^{\mu\nu\rho\sigma\beta\alpha} \\ = & \left((q_A^f)^3 B_{\mu\nu} B_{\rho\sigma} B_{\alpha\beta} + tr F_{\mu\nu} F_{\rho\sigma} F_{\alpha\beta} + 3q_A^f B_{\mu\nu} tr F_{\rho\sigma} F_{\alpha\beta}\right)\epsilon^{\mu\nu\rho\sigma\alpha\beta} \end{aligned} \quad (10)$$

The associated 5-form is easy to compute, although not unique. An obvious choice is [6],[7]

$$\begin{aligned} \omega_5 = & \frac{1}{96\pi^2}[(q_A^f)^3 B_\beta B_{\mu\nu} B_{\rho\sigma} + tr\{A_\beta F_{\mu\nu} F_{\rho\sigma} - \frac{1}{2}A_\beta A_\mu A_\nu F_{\rho\sigma} + \frac{1}{10}A_\beta A_\mu A_\nu A_\rho A_\sigma\} + \\ & 3c_1 tr\{q_A^f B_\beta F_{\mu\nu} F_{\rho\sigma}\} + 3c_2 q_A^f B_{\beta\mu} \omega_{3\nu\rho\sigma}^G]\epsilon^{\beta\mu\nu\rho\sigma} \end{aligned} \quad (11)$$

where $c_1 + c_2 = 1$ and B_μ is the anomalous gauge field. Different choices of c_1, c_2 differ by a total derivative $(\Delta c) d(B\omega_3^G)$. The last term includes the usual Chern-Simons 3-form

$$\omega_3^G = tr\{A_\nu\partial_\rho A_\sigma + \frac{2}{3}A_\nu A_\rho A_\sigma\}\epsilon^{\nu\rho\sigma} \quad (12)$$

whose exterior derivative gives the 4-dimensional Chern character

$$\partial_\mu \omega_{3\nu\rho\sigma}^G \epsilon^{\mu\nu\rho\sigma} = 1/4 F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} \quad (13)$$

and whose gauge variation is

$$\delta_G \omega_3^G = \partial_\nu tr\alpha_G(x)\partial_\rho A_\sigma \quad (14)$$

We can see that this 6-dimensional Chern character does not uniquely define the 5-form ω_5 whose gauge variation yields the consistent Wess-Zumino anomaly in four dimensions, because of the different possibilities for the linear combination of the last two terms subject to the condition $c_1 + c_2 = 1$.

Correspondingly the expression of the solution of the descent equations is not unique. If this solutions were to coincide with divergence of the chiral currents as defined above, we would write after summation over all chiral fermion representations [7]

$$\int \alpha_A(x) \sum_f n_f \partial_\mu J_{Af}^\mu = \frac{1}{24\pi^2} \int \alpha_A(x) \sum_f n_f q_A^f [\frac{1}{4}(q_A^f)^2 B_{\mu\nu} B_{\rho\sigma} + \frac{3}{4}c_1 tr\{\lambda_a \lambda_b\} F_{\mu\nu}^a F_{\rho\sigma}^b] \epsilon^{\mu\nu\rho\sigma} \quad (15)$$

$$\begin{aligned} \int tr\alpha_G(x) \sum_f n_f D_\mu J_{Gf}^\mu = & \frac{1}{24\pi^2} \int \alpha_G^a(x) \sum_f n_f [tr\{\lambda_a \lambda_b \lambda_c\} \partial_\mu (A_\nu^b \partial_\rho A_\sigma^c + \frac{f_{de}^c}{4} A_\nu^d A_\rho^e A_\sigma^e) \\ & + 3/2 c_2 tr\{q_A^f \lambda_a \lambda_b\} B_{\mu\nu} \partial_\rho A_\sigma^b] \epsilon^{\mu\nu\rho\sigma} \end{aligned} \quad (16)$$

It is commonly argued that since the last term in each of the equations (16) can be obtained as the gauge variation of a local counterterm $B_\mu \omega_3^G$, and the effective action can

only be defined in perturbation theory up to such local counterterms, we can set one of the coefficients $c_2 = 0$. This is an obscure statement. A fair attitude is to regard the effective theory as equivalent to a noninvariant classical theory. For a nonanomalous group G , using this choice of coefficients only the mixed and pure $U(1)$ anomalies corresponding to the abelian gauge variation survive and we obtain the results usually cited in the literature [17]. However it is not usually indicated what is the noninvariant action corresponding to this gauge variation.

As we will see, when additional interactions are included in the action to completely cancel the Wess-Zumino anomaly, the term in (16) balanced by a nongauge coupling is that proportional to c_2 . According to the authors of reference [8], in this set of additional interactions only the nongauge coupling is physically relevant. From that point of view it seems more natural to use the mentioned freedom to set the coefficient $c_1 = 0$. A third possibility favored by Bose symmetry in the decomposition of the Chern-Simons form [13] is $c_1 = 1/3, c_2 = 2/3$.

On the other hand if we take the more rigid definition of the anomaly, in the Fujikawa approach such indetermination cannot be allowed. If the quantum theory is to be consistent, either the fermion measure anomalies cancel summing over the fermion content of a theory with a classically gauge invariant action, or they cancel against the gauge variation of some terms in a non-invariant classical action. It can only cancel against the variation of noninvariant terms in the action. The action must be a well defined object with all local interactions clearly specified for a given theory, and in a consistent theory the chiral anomaly cannot admit any indetermination. Only one solution of the descent equations can give the fermion measure anomaly. We will try to determine this below, invoking topological arguments.

It is also interesting to consider the possibility that the nonanomalous factor is an abelian group $U(1)_{NA}$. In this case the descent equations method would start from a Chern character including an additional term

$$\left((q_A^f)^3 B_{\mu\nu} B_{\rho\sigma} B_{\alpha\beta} + (q_{NA}^f)^3 F_{\mu\nu} F_{\rho\sigma} F_{\alpha\beta} + 3q_A^f (q_{NA}^f)^2 B_{\mu\nu} \text{tr} F_{\rho\sigma}^{NA} F_{\alpha\beta}^{NA} + 3(q_A^f)^2 q_{NA}^f B_{\mu\nu} B_{\rho\sigma} F_{\alpha\beta}^{NA} \right) \quad (17)$$

that properly normalized can be obtained as the exterior derivative of the 5-form

$$\omega_5 = \frac{1}{96\pi^2} [(q_A^f)^3 B_\beta B_{\mu\nu} B_{\rho\sigma} + (q_{NA}^f)^3 A_\beta F_{\mu\nu}^{NA} F_{\rho\sigma}^{NA} + 3c_1 \text{tr} q_A^f (q_{NA}^f)^2 B_\beta F_{\mu\nu}^{NA} F_{\rho\sigma}^{NA} + 3c_2 q_A^f (q_{NA}^f)^2 A_\beta B_{\mu\nu} F_{\rho\sigma}^{NA} + 3d_1 B_\beta B_{\mu\nu} F_{\rho\sigma}^{NA} + 3d_2 A_\beta B_{\mu\nu} B_{\rho\sigma}] \epsilon^{\beta\mu\nu\rho\sigma} \quad (18)$$

where as before $d_1 + d_2 = 1 = c_1 + c_2$ and different choices of the coefficients are obtained by adding a total derivative to ω_5 . If the solution to the descent equations gives the gauge variation of the fermion measure, the corresponding gauge anomaly is

$$\int \alpha_A(x) \sum_f n_f \partial_\mu J_{Af}^\mu = \frac{1}{24\pi^2} \int \alpha_A(x) \sum_f n_f q_A^f \left[\frac{1}{4} (q_A^f)^2 B_{\mu\nu} B_{\rho\sigma} + \frac{3c_1}{4} (q_{NA}^f)^2 F_{\mu\nu}^{NA} F_{\rho\sigma}^{NA} + \frac{3d_1}{4} q_A^f q_{NA}^f B_{\mu\nu} F_{\rho\sigma}^{NA} \right] \epsilon^{\mu\nu\rho\sigma} \quad (19)$$

$$\int \alpha_{NA}(x) \sum_f n_f \partial_\mu J_{NAf}^\mu = \quad (20)$$

$$\frac{1}{24\pi^2} \int \alpha_{NA}(x) \sum_f n_f q_{NA}^f \left[\frac{1}{4} (q_{NA}^f)^2 F_{\mu\nu}^{NA} F_{\rho\sigma}^{NA} + \frac{3c_2}{4} q_A^f q_{NA}^f B_{\mu\nu} F_{\rho\sigma}^{NA} + \frac{3d_2}{4} (q_A^f)^2 B_{\mu\nu} B_{\rho\sigma} \right] \epsilon^{\mu\nu\rho\sigma}$$

From our definition of chiral currents, J_{NAf}^μ can be obtained from J_{Af}^μ by replacing $q_A^f \leftrightarrow q_{NA}^f$. Therefore the $U(1)_{NA}$ anomaly functional must be the same as the result of replacing $\alpha_A(x) q_A^f \leftrightarrow \alpha_{NA}(x) q_{NA}^f$ in the $U(1)_A$ anomaly functional. This imposes $c_1 = 1/3 = d_2, d_1 = c_2$ which combined with the constraints $c_1 + c_2 = 1 = d_1 + d_2$ fixes the coefficients to be $d_1 = 2/3 = c_2$ as required by Bose symmetry. We see that the consistent Fujikawa anomaly is completely determined in this case.

As we shall see below, it is not possible to cancel the anomaly terms with a coefficient $\sum_f n_f (q_A^f)^2 q_{NA}^f \neq 0$ using the Green-Schwarz mechanism. Imposing the vanishing of such coefficient for a particular chiral content of the theory many authors do not include these terms in the general expression of the solution of the descent equations for $U(1)_A \times U(1)_{NA}$, and just choose the coefficients $c_1 = 1, c_2 = 0$. We will prove that although this choice is formally incorrect if we want to write the divergence of the usual chiral current, both expressions correspond to the same winding number of the Weyl determinant around a loop of gauge transformations. If such winding number is the relevant physical quantity both results must be considered physically equivalent. Correspondingly we expect the anomaly cancellation conditions resulting from the Green-Schwartz mechanism to be the same.

For G nonabelian, it turns out that none of the solutions of the descent equations presented so far provides the fermion measure anomaly. We will now find the correct solution and prove it to be physically equivalent to (16).

We can try a different solution for the 5-form $\tilde{\omega}_5$, using the general formula [4] for $2n$ dimensions

$$\tilde{\omega}_{2n+1} = (n+1) \int dt t^n \text{tr} \{ A(dA + tA^2)^n \}_{|n=2} = AdAdA + 3/2 AA^2 dA + 3/5 AA^2 A^2 \quad (21)$$

after substitution of $A_\mu = q_A^f B_\mu + A_\mu^a \lambda_a$ for our factor gauge group. The pure $U(1)$ and G terms are the same as in the previous version, but the mixed terms are now very different

$$\begin{aligned} \tilde{\omega}_5 = & \frac{1}{96\pi^2} [(q_A^f)^3 B_\beta B_{\mu\nu} B_{\rho\sigma}] + 4\text{tr} \{ A_\beta \partial_\mu A_\nu \partial_\rho A_\sigma + \frac{3}{2} A_\beta A_\mu A_\nu \partial_\rho A_\sigma + \frac{3}{5} A_\beta A_\mu A_\nu A_\rho A_\sigma \} \\ & + 4\text{tr} q_A^f B_\beta \{ \partial_\mu A_\nu \partial_\rho A_\sigma + 3/2 A_\mu A_\nu \partial_\rho A_\sigma \} + 4\text{tr} q_A^f \partial_\beta B_\mu \{ 2A_\nu \partial_\rho A_\sigma + \frac{3}{2} A_\nu A_\rho A_\sigma \} \} \epsilon^{\beta\mu\nu\rho\sigma} \quad (22) \end{aligned}$$

The exterior derivative of this expression gives the same Chern character as the (11) but the cross terms cannot be obtained from (11) for any choice of c_1, c_2 , and the solution of the consistency conditions that it yields is not the same as (16). The gauge variation of (22) gives a result consistent with the substitution $A_\mu = q_A^f B_\mu + A_\mu^a \lambda_a$ in the general solution of the consistency conditions found by [4] for a simple gauge group

$$\frac{i^n}{(2\pi)^n (n+1)} n(n+1) \int_0^1 dt (1-t) \text{str} \{ \alpha(x) d_x [A(F^t)^{n-1}] \}_{|n=2} = \frac{1}{24\pi^2} \alpha(x) (AdA + \frac{1}{2} AA^2) \quad (23)$$

where $F^t = td_x A + t^2 A^2$. The pure $U(1)_A$ and G anomalies coincide in both solutions but the mixed anomalies are different

$$\int \alpha_A(x) \sum_f n_f \partial_\mu J_{Af}^\mu = \frac{1}{24\pi^2} \int \alpha_A(x) \sum_f n_f q_A^f \left[\frac{1}{4} (q_A^f)^2 B_{\mu\nu} B_{\rho\sigma} + \text{tr} \{ \lambda_a \lambda_b \} \partial_\mu (A_\nu^a \partial_\rho A_\sigma^b + \frac{f_{bde}}{4} A_\nu^a A_\rho^d A_\sigma^e) \right] \epsilon^{\mu\nu\rho\sigma} \quad (24)$$

for a $U(1)_A$ variation and

$$\int \text{tr} \alpha_G(x) \sum_f n_f D_\mu J_{Gf}^\mu = \frac{1}{24\pi^2} \int \alpha_G^a(x) \sum_f n_f [\text{tr} \{ \lambda_a \lambda_b \lambda_c \} \partial_\mu (A_\nu^b \partial_\rho A_\sigma^c + \frac{f_{bde}^c}{4} A_\nu^b A_\rho^d A_\sigma^e) + \text{tr} \{ \lambda_a q_A^f \lambda_b \} \partial_\mu (2B_\nu \partial_\rho A_\sigma^b + \frac{f_{bde}}{4} B_\nu A_\rho^d A_\sigma^e)] \epsilon^{\mu\nu\rho\sigma} \quad (25)$$

for a G variation.

It is easy to see that for $G = U(1)_{NA}$ the substitution $A_\mu = q_A^f B_\mu + q_{NA}^F A_\mu^{NA}$ in the general solution (23) gives the same result as the Bose symmetric choice $d_2 = 1/3 = c_1, c_2 = 2/3 = d_1$ in the previous formulation. This increases our confidence that the prescription to obtain the solution of the descent equations that yields the divergence of the chiral current, is to substitute the gauge connection of the factor group in (23).

For nonabelian G the discrepancy found in both solutions can be understood by realising [9] that the most general solution to the consistency conditions should use the following generalized Chern character

$$\Omega_{2n+2} = c_{n+1} \text{tr} F^{n+1} + \sum_m c_{n,m} \text{tr} F^m F^{n-m} + \sum_{m,l} c_{n,m,l} \text{tr} F^m F^l F^{n-m-l} + \dots \quad (26)$$

This $2n+2$ -form can be written as the exterior derivative of

$$c_{n+1} \tilde{\omega}_{2n+1} + \sum_m c_{n,m} \tilde{\omega}_m \text{tr} F^{n-m} + \dots \quad (27)$$

up to a closed form. In our 4-dimensional $G \times U(1)_A$ case we notice that the the first solution of the consistency conditions is obtained after ignoring the first term in (27) for the calculation of the mixed anomalies. We also notice that the second solution of the consistency conditions only makes use of such term. It was argued in reference [10] that for nonabelian groups in $2n$ dimensions only the term with coefficient c_{n+1} needs to be considered since its result is the one coinciding with perturbative calculations.

Since for our factor gauge group the first two terms in (26) coincide if we set the coefficients equal to one, the corresponding exterior derivatives of the two terms in (27) give the same result. However when we use the Zumino-Stora method to find the divergence of the chiral currents defined above, we should use the first term in (26) as we shall argue, and therefore we should consider the second solution (24)- (25) of the consistency conditions the one providing the fermion measure anomaly.

4 The regularized fermion measure anomaly

To support our assertion that for G nonabelian the second solution of the descent equations is the one that formally reproduces the gauge variation of the fermionic measure , we will

consider now the path integral calculation of the anomaly. We will reproduce for our factor gauge group the analysis of references [11] [14], in which they modify Fujikawa's path integral analysis in the case of chiral fermions. This computation gives the same result as the diagrammatic calculation [5] of the anomaly for a simple nonabelian group, and that fact leads to the belief that the regularization used by [11] is correct.

For a chiral fermionic action

$$i\bar{\psi} \not{D} 1/2(1 + \gamma_5)\psi = \bar{\psi}\gamma^\mu(\partial_\mu + A_\mu)1/2(1 + \gamma_5)\psi, \quad A_\mu = q_A^f B_\mu + A_\mu^a \lambda_a \quad (28)$$

the Fujikawa method requires extending the fermion sector to gauge invariant righthanded partners, that only introduce a numerical factor after their integration in the path integral. The total fermionic action is thus

$$\bar{\psi} \not{\tilde{D}} \psi = \bar{\psi}\gamma^\mu \left(\partial_\mu + A_\mu \frac{1 + \gamma_5}{2} \right) \psi = \bar{\psi}\gamma^\mu \left(\partial_\mu \frac{1 - \gamma_5}{2} + [\partial_\mu + A_\mu] \frac{1 + \gamma_5}{2} \right) \psi \quad (29)$$

where the operator $\not{\tilde{D}}$ has well defined eigenvalues as opposed to the original \not{D} that maps positive chirality spinors to negative chirality ones. We can use these eigenvalues to construct a Fujikawa type regulator for the anomaly.

Working in Euclidean space-time after a Wick rotation of the action, the authors in [11] expand the Dirac fermion operators in a basis of eigenstates of the modified Dirac operator (which being nonhermitian forces us to consider both eigenstates of $i \not{\tilde{D}}$; and $(i \not{\tilde{D}})^\dagger$)

$$\begin{aligned} \psi(x) &= \sum_n a_n \phi_n(x) \quad , \quad \bar{\psi}(x) = \sum_n \bar{b}_n \chi_n^\dagger(x) \\ i \not{\tilde{D}} \phi_n(x) &= k_n \phi_n(x) \quad , \quad (i \not{\tilde{D}})^\dagger \chi_n(x) = k_n^* \chi_n(x) \end{aligned}$$

where the coefficients a_n, \bar{b}_n are Grassman valued. The commuting Lorentz spinors $\phi_n(x)$ and $\chi_n(x)$ obey orthonormality and closure relations

$$\int d^4x \chi_m^\dagger(x) \phi_n(x) = \delta_{m,n} \quad ; \quad \sum_n \phi_n^{\beta,p}(x) \chi_n^{\dagger \gamma,q}(y) = \delta^{\beta,\gamma} \delta^{p,q} \delta^4(x - y) \quad (30)$$

After a gauge transformation the transformed Grassman coefficients can be obtained as

$$\begin{aligned} a'_n &= \int d^4x \chi_n^\dagger(x) (\psi^{L'}(x) + \psi^R(x)) = \\ &= \int d^4x \sum_m \left(a_m \chi_n^\dagger(x) \exp(-n_L \alpha(x) \gamma_5) P_L \phi_m(x) + a_m \chi_n^\dagger(x) P_R \phi_m(x) \right) \\ &= \sum_m (C_{n,m}^L + \delta_{n,m}^R) a_m \end{aligned} \quad (31)$$

and similarly for the transformed \bar{b}_n

Rewriting the path integral fermionic measure in terms of the grassman coefficients it is very simple to find the Jacobian

$$\prod_x D\bar{\psi}^{L'}(x) D\psi^{L'}(x) = \prod_m d\bar{b}'_m \prod_n da'_n = (\det \bar{C})^{-1} \prod_m d\bar{b}_m (\det C)^{-1} \prod_n da_n \quad (32)$$

For an infinitesimal transformation we can evaluate $\det C = \exp(\text{tr} \log C)$ by expanding C to first order in α so the jacobian can also be written to first order in the transformation parameter

$$(\det C)^{-1} = \exp(\text{tr} \int d^4x \phi_m^{L\dagger}(x) n_L \alpha(x) \gamma_5 P_L \phi_n(x)) \simeq 1 + \sum_n \int d^4x \chi_n^\dagger(x) n_L \alpha(x) \gamma_5 P_L \phi_n(x) \quad (33)$$

Similarly the we find the Jacobian resulting from the transformation of the Dirac conjugate fermion and combining both

$$\begin{aligned} (\det C^R)^{-1} (\det C^L)^{-1} &= \exp\{\text{tr} \int d^4x \chi_m^\dagger(x) n_L \alpha(x) \gamma_5 \phi_n(x) \\ &\simeq 1 + \sum_n \int d^4x \chi_n^\dagger(x) n_L \alpha(x) \gamma_5 \phi_n(x) \end{aligned} \quad (34)$$

The closure relations that the basis ϕ_n, χ_n obeys reduce the anomaly of the lefthanded fermion field to

$$\int d^4x \alpha(x) A(x) = \int d^4x \alpha^a(x) n_f \text{tr}\{T_a \gamma_5 \delta^4(x-x)\} \quad (35)$$

Although $\text{tr} \gamma_5 = 0$ the anomaly as it stands is an ill-defined quantity because of the divergence introduced by the delta function. To regularize á la Fujikawa we need to include a weight factor $\exp((-i \not{D})^2/M^2)$ that reduces to one in the limit $M \rightarrow \infty$. With this regularization, we expand the eigenstates of the modified Dirac operator on plane waves

$$\begin{aligned} \alpha(x) A(x) &= \lim_{x \rightarrow y, M \rightarrow \infty} \sum_n \phi_n^\dagger(y) g_a \alpha^a(x) T_a \gamma_5 \exp\left(-(\not{D})^2/M^2\right) \phi_n(x) = \\ &= \lim_{x \rightarrow y, M \rightarrow \infty} \alpha^a(x) \int \frac{d^4k}{(2\pi)^4} e^{-iky} \text{tr} T_a \gamma_5 \exp\left(-(\not{D})^2/M^2\right) e^{ikx} \end{aligned} \quad (36)$$

Rewriting the operator $(\not{D})^2$ as in reference [11]

$$(\not{D})^2 = (\not{\partial} P_R + \not{D} P_L)^2 = -(\not{\partial} \not{D} P_L + \not{D} \not{\partial} P_R) \quad (37)$$

and decomposing the chirality operator $\gamma_5 = P_L - P_R$ we can separate the regularized anomaly

$$\text{tr} T_a \gamma_5 \exp\left(-(\not{D})^2/M^2\right) = \text{tr} T_a P_L \exp(\not{\partial} \not{D}/M^2) - \text{tr} T_a P_R \exp(\not{D} \not{\partial}/M^2) \quad (38)$$

Pulling the plane wave factor to the left, we expand the exponential around $-k^\mu k_\mu$ in both terms. After integration on k the only abnormal parity terms that survive in the limit $M \rightarrow \infty$ are those proportional to M^{-4} . The result of adding the contributions from both terms is [11] the consistent anomaly

$$\frac{1}{24\pi^2} \int d^4x \text{tr} n_L \{\alpha(x) \partial_\mu (A_\nu \partial_\rho A_\sigma + \frac{1}{2} A_\nu A_\rho A_\sigma)\} \epsilon^{\mu\nu\rho\sigma} \quad (39)$$

After substituting $A_\mu = q_A^f B_\mu + A_\mu^a \lambda_a$ and $\alpha = \alpha_A + \alpha_a \lambda_a$, we obtain the gauge anomaly by summing the contributions of all lefthanded ($n_L = +1$) and righthanded ($n_R = -1$) fermion representations of $G \times U(1)_A$. This is precisely the second solution (24)-(25) of the descent equations.

5 Topological equivalence of different solutions of the descent equations

Now that we have found a regularization for the anomalous variation of the fermion measure that supports the alternative solution of the descent equations presented in this paper, further study of the connection between the Zumino-Stora method and the consistent anomaly is needed to understand why both solutions are physically equivalent.

For that purpose, it is necessary to explain why a solution of the descent equations yields the same result as the local density resulting from the gauge variation of the path integral fermionic measure, for a chiral theory in $2n$ dimensions. This mysterious coincidence has been explained by Alvarez-Gaumé and Ginsparg by noticing that the anomaly functional can be understood as an infinitesimal gauge variation in the phase of the Weyl determinant that formally defines the fermionic effective action (the norm of such determinant is gauge invariant)

$$\int D\bar{\psi}D\psi \exp\left(\int dx i\bar{\psi} \tilde{\mathcal{D}}(A) \psi\right) = \exp \Gamma(A) = \det i \tilde{\mathcal{D}}(A) = |\det i \tilde{\mathcal{D}}(A)| e^{iw(A,\theta)} \quad (40)$$

Using topological arguments, the integrated gauge variation of the Weyl determinant, i.e. its winding number, is proved to be equivalent to the index of the Dirac operator [11] in a $2n+2$ -dimensional space constructed by tensoring the $2n$ -dimensional Euclidean space (compactified to a $2n$ -sphere) with a disc parametrized by polar coordinates t and θ . In this extended space gauge connections are defined as

$$A(t, \theta)_\mu = tA_\mu^\theta = tg^{-1}(\theta, x)(A_\mu + \partial_\mu)g(\theta, x) \quad (41)$$

$$A_\theta = tg^{-1}(\theta, x)\partial_\theta g(\theta, x), \quad A_t = 0 \quad (42)$$

$$g(0, x) = 1 = g(2\pi, x) \quad (43)$$

so that the coordinate θ parametrizes a path of gauge transformed connections on the border of the disc. Infinitesimal displacements $\theta \rightarrow \theta + \delta\theta$ along the path can be seen as additional gauge transformations with gauge parameter $\alpha = g^{-1}d_\theta g$.

When we integrate the gauge variation of the effective action over a loop of gauge transformed configurations we find the winding number for the phase of the Weyl determinant, which coincides with the index of the Dirac operator on the disc

$$\int_0^{2\pi} d\theta \delta_\theta \Gamma(A^\theta) = \int_0^{2\pi} d\theta \frac{\partial w(A, \theta)}{\partial \theta} = \text{ind}(i \mathcal{D}_{2n+2})|_{S^{2n} \times \text{Disc}} \quad (44)$$

To apply the index theorem on this $2n+2$ -dimensional space bounded by the edge of the disk, boundary conditions are implicitly given by mapping the disk to the upper patch of a $2n+2$ -sphere and defining gauge connections on a lower patch covering the rest of the sphere. Parametrizing the distance to the pole on each patch with $0 < t < 1$ and $0 < s < 1$ the boundary between lower and upper patches $t = 1 = s$ is identified with the edge of the disk, and the gauge connection is defined on this lower patch as

$$A(s, \theta)_\mu = A_\mu^\theta \quad (45)$$

$$A_\theta = 0 = A_s \quad (46)$$

so that the transition function on the boundary between patches is precisely $g(\theta, x)$. The corresponding extended field strength is then constructed on the upper patch

$$\mathcal{F} = (d_x + d_t + d_\theta)\mathcal{A}(t, \theta) + \mathcal{A}^2(t, \theta) = F_x(t, \theta) + td_x\alpha + A^\theta dt + \alpha dt + td_\theta A^\theta \quad (47)$$

$$F_x(t, \theta) = td_x A^\theta + t^2(A^\theta)^2, F_{t,\mu} = A_\mu^\theta, F_{\theta\mu} = (t^2 - t)[\alpha, A_\mu^\theta] +, F_{t\theta} = \alpha \quad (48)$$

and the lower patch

$$\mathcal{F} = (d_x + d_s + d_\theta)\mathcal{A}(s, \theta) + \mathcal{A}^2(s, \theta) = F_x \quad (49)$$

Using the Atiyah-Singer index theorem we can reproduce the index of the Dirac operator by integrating the Chern character constructed from this generalized field strength over the $(2n+2)$ -dimensional space. Since the Chern character is an exact $2n+2$ -form, the result of this integration is only the difference between the Chern-Simons $2n+1$ -forms of the lower and upper patches evaluated at the boundary $t = 1 = s$:

$$\begin{aligned} \text{ind}(i \not{D}_{2n+2}) &= \frac{i^{n+1}}{(2\pi)^{n+1}(n+1)!} \int_{S^2 \times S^{2n}} \text{tr} \mathcal{F}^{n+1} = \\ &= \frac{i^{n+1}}{(2\pi)^{n+1}(n+1)!} \int_{S^1 \times S^{2n}} d\theta d^{2n}x (\omega_{2n+1}(t=1) - \omega_{2n+1}(s=1)) \end{aligned} \quad (50)$$

Since the Chern-Simons form on the lower patch does not contain any $d\theta$ or ds its contribution vanishes. On the upper patch we need only to consider the component of the Chern-Simons form with no t lower index. This component is linear in $\alpha(x)$ because the form can only contain one $d\theta$ differential. The gauge parameter and the factor it multiplies can be written as a θ exterior derivative, precisely the gauge variation of the Chern-Simons form as prescribed by the Zumino-Stora method

$$\omega_{2n+1} = \omega_{2n+1}^0 + \alpha d\theta \wedge \omega_{2n}^1, \delta \omega_{2n+1} = d\omega_{2n}^1 \quad (51)$$

We arrive at two conclusions from this analysis. First, when we apply the Zumino-Stora method to find the divergence of the chiral currents defined above, i.e. the fermion measure anomaly, we should only consider the first term in (26). As a plus we get the normalization factor of the nonabelian anomaly from the index theorem normalization constant [11].

The second conclusion is that we have found a characterization of physical equivalence for the theories we consider. Two solutions of the descent equations are physically equivalent if the corresponding Chern-Simons 5-forms belong to the same cohomology class. When we integrate these forms over $S^1 \times S^{2n}$ we find the same result. As we have seen this amounts to integrate the infinitesimal gauge transformation of the action over a loop of such transformations. The local counterterms that make the Lagrangian noninvariant do not contribute to this loop, and all the theories in the same class have the same integrated

anomaly. It is always equal to the winding number of the Weyl determinant, i.e. the integrated fermion measure anomaly.

If two Chern-Simons 5-forms differ by a closed form, when we integrate over the parameter θ we will not in general find the same winding number for the phase of the Weyl determinant, and we cannot consider the corresponding expressions for the anomaly density physically equivalent. Different Weyl determinant winding numbers correspond to different fermion measure anomalies, but if the expression of this consistent anomaly is truly regularization independent, this situation is not possible for a consistent theory.

In the case we are studying, the difference between both versions of the Chern-simons form can be straightforwardly computed

$$\begin{aligned}\tilde{\omega}_5 - \omega_5 = & Btr(dAdA + \frac{3}{2}A^2dA) + dBtr(2dAdA + \frac{3}{2}AA^2) - 3c_1Btr(dAdA + 2A^2dA) \\ & - 3c_2dBtr(AdA + \frac{2}{3}AA^2) = (1 - 3c_1)Btr(dAdA) + (\frac{3}{2} - 6c_1)Btr(A^2dA) + \\ & (2 - 3c_2)dBtr(AdA) + (\frac{3}{2} - 2c_2)dBtr(AA^2)\end{aligned}\quad (52)$$

Terms with two derivatives can only be obtained from $tr(BAdA)$ because $tr(dBA^2) = 0$, while terms with one derivative can only come from differentiating $trBAA^2$. It is easy to see that only $c_1 + c_2 = 1$ can simultaneously obey $1 - 3c_1 = -(2 - 3c_2)$ and $3/2 - 6c_1 = -3(3/2 - 2c_2)$, converting the expression above into a total derivative.

Therefore, we learn that although the second prescription for the consistent gauge anomaly is the one that we would find by proper regularization of the Fujikawa method, the first solution of the descent equations, while representing a different density does not change the global winding number of the weyl determinant for any choice of the coefficients $c_1 + c_2 = 1$. It seems that the relevant physical information carried by the anomaly is contained in this topological quantity.

It is worth remarking that this justification of the use of the descent equations only works for a very particular choice in reference [12] of the extended gauge connections on the two patches of the $2n+2$ -dimensional space. A different but also acceptable choice in references [11] and [13] leads directly to the general solution (23) and does not allow to make contact with the gauge variation of the Chern-Simons $2n+1$ -forms as prescribed by the descent equations.

6 The Green-Schwarz mechanism and anomaly cancellation conditions

We have mentioned that the fermion measure anomaly is the only piece in the gauge variation of the effective action that cannot be balanced by the gauge variation of local terms in gauge fields. If it does not vanish when summing its coefficient over all chiral fermion representations, the theory is in principle unacceptable as a quantum theory. Nevertheless, in certain cases when the Chern character factorizes, it is possible to balance this nonzero anomaly against the special gauge variation of a bosonic degree of freedom coupled to gauge fields [3] and some local counterterms. This is known as the Green-Schwarz mechanism. Once the fermion measure anomaly is cancelled by such local terms, any additional piece

in the gauge variation of the effective action can be cancelled by the gauge transformation of gauge field counterterms as we know. Therefore even if we use different solutions of the descent equations as anomaly densities of physically equivalent actions, the constraint on the physical spectrum of particles that the cancellation imposes must be the same.

To test the equivalence of different anomaly expressions, let us study the anomaly cancellation conditions derived from implementing the 4-dimensional Green-Schwarz mechanism. For completeness, we will consider the possibility of having mixed gravitational- $U(1)_A$ anomalies. We would compute them from a similar method, but using the product $(-1/8)TrFtrRR$ in the descent equations instead [12]. Pure gravitational anomalies do not exist in four dimensions but for $4n + 2$ dimensions they would be computed from $trRRR$ using a generalization of the arguments that justify the use of the descent equations in the pure gauge case [12]. The mixed gravitational- $U(1)_A$ anomaly arising from a $U(1)_A$ variation of the path integral measure is uniquely determined

$$-\frac{1}{24\pi^2} \int \alpha_A(x) \sum_f n_f q_A^f \frac{1}{8 \times 4} tr R_{\mu\nu} R_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} \quad (53)$$

Assuming that the gauge group G is anomaly free $\sum_f tr \lambda_a \lambda_b \lambda_c = 0$, complete cancellation of the anomaly (16) through the four dimensional Green-Schwarz mechanism, requires the mixed gauge, pure $U(1)_A$ gauge and the mixed gravitational- $U(1)_A$ anomalies to balance the gauge variation of the additional couplings introduced in the field theory action

$$S \rightarrow S - \frac{1}{24\pi^2} \int \left(\frac{c_A \Lambda}{2g_A} M_{\mu\nu} B_{\rho\sigma} + \frac{c_G 3c_1}{g_A g_G^2} B_\mu \omega_{3\nu\rho\sigma}^G - \frac{c_L}{g_A g_L^2} B_\mu \omega_{3\nu\rho\sigma}^L \right) \epsilon^{\mu\nu\rho\sigma} \quad (54)$$

The gauge fields are normalized in this couplings so that they do not contain the coupling constant implicit in the covariant derivative. Consistent with this normalization the spin connection is also formally normalized by a coupling constant to be defined later by analogy with the gauge coupling constants, although it does not have the same physical meaning.

The field M is the bosonic, second rank tensor in the supergravity multiplet that couples to $tr F^2$ in $2n$ dimensions. Such coupling is a one loop string effective term [8] not present in the tree level action derived from the low energy limit of the string. We have also the mass parameter $\Lambda = \sqrt{2}/\kappa$ defined from the field strength of B [3], for our factor gauge group

$$H = dB + \frac{\kappa}{\sqrt{2}} \left(\frac{\omega_3^A}{g_A^2} + \frac{\omega_3^G}{g_G^2} - \frac{\omega_3^L}{g_L^2} \right) \quad (55)$$

By analogy with 10-dimensional supergravity [15], this field strength is defined to include Chern-Simons gauge forms normalized by the gauge coupling constants. To achieve anomaly cancellation we need to extend the field strength to include also Lorentz Chern-Simons forms coming from a higher order string effective term [3]. It is common to use the same form of normalization as the one appearing in the gauge forms. In 10-dimensional string inspired theories, since the anomaly free gauge groups considered have a unique coupling constant, such normalization is well defined [3]. In the 4-dimensional analog different gauge factors have different gauge couplings at tree level given by $1/g_G^2 = k_G/g^2$ [16] where k_G is the Kac level of G and g^2 is the v.e.v. of the dilaton. It is not clear what normalization should be used for the Lorentz Chern-Simons form unless the coefficient of this effective term is

computed from the 4-dimensional string theory considered. We will formally define a “Kac level” for gravity $1/g_L^2 = k_L/g^2$. It is only a normalization factor defining the gravitation term in (55), in principle having no relation with any Kac-Moody algebra.

Gauge invariance of the field strength H imposes a nontrivial gauge (and Lorentz) variation of the antisymmetric supergravity tensor

$$\delta M_{\mu\nu} = \left(\frac{1}{\Lambda g_A^2} \alpha_A \partial_\mu B_\nu + \frac{1}{\Lambda g_G^2} \text{tr} \{ \alpha_G \partial_\mu A_\nu \} + \frac{1}{\Lambda g_L^2} \text{tr} \{ \Delta_L \partial_\mu \omega_\nu^L \} \right) \epsilon^{\mu\nu} \quad (56)$$

where ω_μ^L is the spin connection.

The cancellation of the anomaly generated by a $U(1)$ transformation fixes the coefficients of the added couplings to be

$$c_A = g_A^3 \sum_f n_f (q_A^f)^3 ; \quad c_G = g_A g_G^2 \sum_f n_f \text{tr} q_A^f \lambda_a \lambda_b ; \quad c_L = \frac{g_A g_L^2}{8} \sum_f n_f q_A^f \quad (57)$$

The last sum runs over all chiral fermions with anomalous charges since all of them couple to gravity.

Cancellation of the mixed anomaly generated by a G transformation is possible only if

$$\frac{c_A}{2g_A g_G^2} - \frac{3c_1 c_G}{2g_A g_G^2} = 3/2 c_2 \sum_f n_f \text{tr} q_A^f \lambda_a \lambda_b \quad (58)$$

The additional condition $c_1 + c_2 = 1$ imposes a nontrivial constraint on the anomalous charges of the chiral fermion content of the theory

$$g_A^2 \sum_f n_f (q_A^f)^3 = 3g_G^2 \sum_f n_f \text{tr} q_A^f \lambda_a \lambda_b \quad (59)$$

Similarly, the absence of any anomaly under Lorentz transformations imposes

$$g_A^2 \sum_f n_f (q_A^f)^3 = \frac{g_L^2}{8} \sum_f n_f q_A^f \quad (60)$$

The cancellation condition we have found from the Lorentz transformation is not the usual one found in the literature . The reason for this is that some of the anomaly cancelling terms are usually transformed to a dual version of the theory, in which they are Lorentz invariant, while the rest of the terms are forgotten. It is the omitted terms the ones that impose this last condition. We will come back to this point later when the dual version is studied.

An argument supporting the cancellation conditions (60) is the coefficient of the MdB coupling obtained from the string theory 1-loop amplitude [8] , which seems to be

$$g_A^{-2}/48\pi^2 \sum_f n_f q_A^f \quad (61)$$

Unless $\sum_f n_f q_A^f = 2 \sum_f n_f (q_A^f)^3$, which is precisely our condition (60) when $k_L = 4k_A$, we will not have the right coefficient for the MdB term. This relation between anomalous charges differs by a factor of four from the actual relation found in the particular model of

reference [18]. It might be due to a different normalization convention of the field H used by [8] in their computation of (61).

Now we must make sure it is possible to remove the anomalies in our second solution of the descent equations (24)- (25) when $\sum_f n_f \text{tr}\{\lambda_a q_A^f \lambda_b\} \neq 0 \neq \sum_f q_A^f q_A^f q_A^f$, by a 4-dimensional Green-Schwarz mechanism similar to that discussed for the first solution of the descent equations. The cancellation conditions should be the same in both cases.

Following the path we tried before, we can try adding to the classical action the coupling of the supergravity second rank tensor with the anomalous field strength, and the coupling of the anomalous gauge field with the three form that defines the mixed $U(1)_A$ anomaly. (since the mixed gravitational- $U(1)_A$ anomaly has not changed, the cancellation term and resulting condition are the same as before)

$$S \rightarrow S - \frac{1}{24\pi^2} \int \left(\frac{c_A}{g_A} \Lambda M_{\mu\nu} \partial_\rho B_\sigma + \frac{c_G}{g_A g_G^2} B_\mu \tilde{\omega}_3^G{}_{\nu\rho\sigma} - \frac{c_L}{g_A g_L^2} B_\mu \omega_3^L{}_{\nu\rho\sigma} \right) \epsilon^{\mu\nu\rho\sigma} \quad (62)$$

Since the second solution of the descent equations is the one corresponding to the fermion measure anomaly, the Green-Schwarz terms that we are adding are precisely the effective couplings that complete the gauge invariant action to make an anomaly free theory.

The 3-form

$$\tilde{\omega}_3^G = \text{tr}\{A_\nu \partial_\rho A_\sigma + \frac{1}{2} A_\nu A_\rho A_\sigma\} \epsilon^{\nu\rho\sigma} = (A_\nu^a \partial_\rho A_\sigma^a + \frac{1}{4} f_{ade} A_\nu^a A_\rho^d A_\sigma^e) \epsilon^{\nu\rho\sigma} \quad (63)$$

has the following G gauge variation

$$(2\partial_\nu \alpha^a(x) \partial_\rho A_\sigma^a - \frac{f_{ade}}{4} \partial_\nu \alpha^a(x) A_\rho^d A_\sigma^e) \epsilon^{\nu\rho\sigma} \quad (64)$$

It is easy to see that after partial integration, the $U(1)_A$ and G gauge variations of (62) can cancel the gauge and mixed gravitational-gauge anomalies we found provided

$$g_A^3 \sum_f n_f q_A^f q_A^f q_A^f = c_A \quad (65)$$

$$g_A g_G^2 \sum_f n_f \text{tr}\{q_A^f \lambda_a \lambda_b\} = c_G \quad (66)$$

$$g_A g_G^2 \sum_f n_f \text{tr}\{\lambda_a q_A^f \lambda_b\} = c_A - 2c_G \quad (67)$$

The nontrivial constraint on the spectrum of fermion anomalous charges $3c_G = c_A$ arises again as expected. The vanishing Lorentz variation of (62) imposes the same constraint as before.

Using the string tree level definition of the gauge coupling constants an interesting consequence of the cancellation conditions appears if the nonanomalous factor G is a factor group itself $G = G_1 \times G_2$. The constraints we have found impose that the sum of anomalous charges for the spectrum of fermions transforming under each subfactor is proportional to the corresponding Kac level [21]

$$\sum_f n_f q_A^f \text{tr} \lambda_a^{G_1} \lambda_b^{G_1} / \sum_f n_f (q_A^f)^3 = k_a / 3k_A \quad (68)$$

As promised before we will perform the analysis of the Green-Schwarz mechanism for the case $G = U(1)_{NA}$. We could try to include the following additional anomaly cancelling terms in the action

$$\frac{1}{24\pi^2} \int \left(\frac{c_{NA}}{g_{NA}} \Lambda M_{\mu\nu} \partial_\rho A_\sigma + \frac{c_G}{g_{NA} g_A^2} A_\mu \text{tr} \tilde{\omega}_{3\nu\rho\sigma}^A \right) \epsilon^{\mu\nu\rho\sigma} \quad (69)$$

but the first term introduces a non vanishing $U(1)_{NA}$ variation without a corresponding pure $U(1)_{NA}$ anomaly to cancel, therefore the coefficient c_{NA} must be zero. It is easy to see that the second term cannot cancel by itself the mixed anomalies in (20). Another nontrivial constraint arises again

$$\sum_f n_f (q_A^f)^2 q_{NA}^f = 0 \quad (70)$$

This result agrees with the cancellation conditions found in the literature for such mixed abelian anomalies.

7 Dual version of the Green-Schwarz terms and axion-like couplings

As a final exercise, we will now review how to rewrite the Green-Schwarz terms in the action so that a coupling of a pseudoscalar to the 4-dimensional Chern character appears in the effective field theory. First we consider the case when the anomaly is given by the factorized solutions of the descent equations [17], and then we will reproduce the analysis for the second form of the anomaly presented in this paper, showing that the same pseudoscalar coupling is found.

Since the antisymmetric tensor field $M_{\mu\nu}$ only contains one degree of freedom [20], it is common in the literature to replace it by a pseudoscalar through a duality transformation. Again following reference [17], which provides the most elegant explanation, we can see that the field $M_{\mu\nu}$ appears on the kinetic term $H^* H$ containing its gauge and Lorentz invariant field strength (55) that obeys a generalized Bianchi identity

$$dH = \frac{1}{4\Lambda} \left(\frac{F_A^2}{g_A^2} + \frac{\text{tr} F_G^2}{g_G^2} - \frac{\text{tr} R^2}{g_L^2} \right) \quad (71)$$

and it also appears in the Green-Schwarz term $\frac{c_A \Lambda}{g_A} M dB^A$. The field equation of $M_{\mu\nu}$

$$-\partial_\rho H^{\rho\mu\nu} + \frac{c_A \Lambda}{48\pi^2 g_A} \partial_\rho B_\tau \epsilon^{\mu\nu\rho\tau} = 0 \quad (72)$$

allows us to write the field strength in terms of the divergence of a pseudoscalar for $M_{\mu\nu}$ on shell

$$-H^{\rho\mu\nu} + \frac{c_A \Lambda}{48\pi^2 g_A} B_\tau \epsilon^{\mu\nu\rho\tau} = \partial_\tau \theta(x) \epsilon^{\nu\mu\rho\tau} \quad (73)$$

This relation provides an alternative description of theory when we use it to rewrite the kinetic term $[*d\theta - 1/(48\pi^2 g_A) c_A \Lambda^* B][d\theta - 1/(48\pi^2 g_A) c_A \Lambda B]$ and the first of the Green-Schwarz terms

$$\frac{c_A \Lambda}{24\pi^2 g_A} \int M_{\mu\nu} \partial_\rho B_\sigma \epsilon^{\mu\nu\rho\sigma} = \frac{c_A \Lambda}{24\pi^2 g_A} \int M_{\mu\nu} \frac{48\pi^2 g_A}{c_A \Lambda} \partial_\rho H^{\rho\mu\nu} \quad (74)$$

partial integrating we find

$$\begin{aligned} -2 \int \partial_\rho M_{\mu\nu} H^{\rho\mu\nu} &= 2 \int \left(H_{\rho\mu\nu} - \frac{1}{\Lambda} (g_A^{-2} \omega_3^A + g_G^{-2} \omega_3^G - g_L^{-2} \omega_3^L)_{\rho\mu\nu} \right) H^{\rho\mu\nu} = \\ 2 \int \left(-H_{\rho\mu\nu} H^{\rho\mu\nu} + \frac{1}{\Lambda} (g_A^{-2} \omega_3^A + g_G^{-2} \omega_3^G - g_L^{-2} \omega_3^L)_{\rho\mu\nu} (\partial_\tau \theta(x) - \frac{c_A \Lambda}{48\pi^2 g_A} B_\tau) \epsilon^{\tau\rho\mu\nu} \right) \end{aligned} \quad (75)$$

The first term combines with the kinetic term, while the second one and the additional piece in the original anomaly cancelling term (54) provide the new cancellation terms in this description of the theory

$$\begin{aligned} 2 \int \left[-\frac{\theta(x)}{\Lambda} \partial_\tau (g_A^{-2} \omega_3^A + g_G^{-2} \omega_3^G - g_L^{-2} \omega_3^L)_{\rho\mu\nu} - \frac{1}{24\pi^2 g_A} c_A B_\tau (g_G^{-2} \omega_3^G - g_L^{-2} \omega_3^L)_{\rho\mu\nu} + \right. \\ \left. \frac{1}{24\pi^2 g_A} B_\tau (3c_1 c_G g_G^{-2} \omega_3^G - c_L g_L^{-2} \omega_3^L)_{\tau\mu\nu} \right] \epsilon^{\tau\rho\mu\nu} \end{aligned} \quad (76)$$

where the antisimetrization of Lorentz indices has made the term $\omega_3^A B$ vanish. The first three terms give the axion-like couplings of the pseudoscalar

$$2 \int \frac{\theta(x)}{g^2 \Lambda} 1/4 (k_A \text{tr} B_{\mu\nu} B_{\rho\sigma} + k_G \text{tr} F_{\mu\nu} F_{\rho\sigma} - k_L \text{tr} R_{\mu\nu} R_{\rho\sigma}) \epsilon^{\mu\nu\rho\sigma} \quad (77)$$

Now the equation of motion of the pseudoscalar contains the Bianchi identity of the field strength H , while the equation of motion of the tensor field M (72) can be seen as a Bianchi identity for $\theta(x)$ [17]. This is therefore a dual description of the theory.

To see how the anomaly cancellation happens in the dual model we notice that the pseudoscalar must shift under a $U(1)_A$ tranformation to maintain the gauge invariance of H in (73)

$$\delta_A \theta(x) = \frac{c_A \Lambda}{48\pi^2 g_A} \alpha_A(x) \quad (78)$$

This gauge shift under $U(1)_A$ is precisely the one needed for the terms in (76) to cancel the anomalous terms in (15). Under a G gauge variation we obtain anomaly cancellation provided the condition $c_A - 3c_1 c_G = 3c_2 c_G$ is met again.

In this formulation, the duality transformation apparently simplifies the theory when $c_1 = 1, c_2 = 0$. In that case the cancelling terms are just the axion-like couplings, all the anomalies are generated by $U(1)_A$, and the gauge tranformation (78) achieves their complete cancellation. However, we have already seen that such a choice amounts to include compensating local counterterms together with the minimal action, that make the total action noninvariant. It is common in the literature to make the choice of coefficients mentioned

and forget the additional terms in the action. Without them the minimal action plus dualised axion-like Green-Schwarz terms are invariant under Lorentz transformations, and the nontrivial constraint (60) is never found.

The cancellation condition involving $\sum_f n_f q_A^f$ that can be usually found in the literature, assumes that the last term in (77) is the one responsible for the cancellation of the mixed $U(1)_A$ -gravitational anomaly. In that case we would find a constraint similar to (68), i.e. the ratio of the gravitational “Kac” level and the anomaly coefficient $\sum_f n_f q_A^f$ should be commensurate to the corresponding ratios for the gauge factors.

$$\frac{3 \sum_f n_f q_A^f \text{tr} \lambda_a \lambda_b}{k_a} = \frac{\sum_f n_f (q_A^f)^3}{K_A} = \frac{\sum_f n_f q_A^f}{8k_L} \quad (79)$$

We can see that including all the Green-Schwarz terms as we should, the $U(1)_A$ gauge variation of the second term in (76) balances the variation of the axion-like couplings θF^2 and θR^2 in the first term, so that the anomaly cancellation is provided by the third term. Therefore we recover the correct cancellation condition (57) by keeping all the Green-Schwarz terms.

Keeping these additional local counterterms in the dual version of the action, we can still identify the pseudoscalar $\theta/g^2\Lambda$, with the imaginary part of the scalar component of the chiral superfield that defines the gauge coupling. In supersymmetric theories this coupling is given by the dilaton v.e.v. multiplying the gauge kinetic function (up to a global normalization factor)

$$f_{ab}(S)W_\alpha^a W^{b\alpha} |_{\theta\theta} = k_a(S + S^\dagger)\delta_{ab}F_{\mu\nu}^a F^{b\mu\nu} + k_a(S - S^\dagger)\delta_{ab}F_{\mu\nu}^a \tilde{F}^{b\mu\nu} \simeq \frac{1}{g_a^2}F_{a\mu\nu}F_a^{\mu\nu} + k_a\frac{\theta(x)}{\Lambda'}F_{a\mu\nu}\tilde{F}_a^{\mu\nu} \quad (80)$$

From our identification, the mass scale $\Lambda' = g^2/\Lambda$. The axion-like coupling $\theta \text{tr} R\tilde{R}$ is not present in minimal supersymmetric theories, but it is usually understood as a higher order string effect.

Performing the same duality transformation on the Green-Schwarz terms that cancel the fermion measure anomaly (25), we can see that we obtain again axion-like couplings. In this case we have an explicit expression of the complete anomaly free action, and the noninvariant terms cannot be accidentally omitted.

The field equation of M and the definition of the pseudoscalar are the same as before. Rewriting the Green-Schwarz interactions (62) in terms of such pseudoscalar we find

$$\frac{1}{24\pi^2} \int \left(\frac{c_A \Lambda}{g_A} M_{\mu\nu} \partial_\rho B_\sigma + \frac{c_G}{g_A g_G^2} B_\mu \text{tr} \tilde{\omega}_{3\nu\rho\sigma}^G \right) \epsilon^{\mu\nu\rho\sigma} = -2 \int H_{\rho\mu\nu} H^{\rho\mu\nu} + \int \left[-2 \frac{\theta(x)}{g^2 \Lambda} \partial_\tau (k_A \omega_3^A + k_G \omega_3^G - k_L \omega_3^L)_{\rho\mu\nu} + \frac{1}{24\pi^2 g_A} B_\tau (c_G g_G^{-2} \tilde{\omega}^G - c_A g_G^{-2} \omega_3^G + (c_A - c_L) g_L^{-2} \omega_3^L)_{\rho\mu\nu} \right] \epsilon^{\mu\nu\rho\sigma} \quad (81)$$

We find again the coupling of the pseudoscalar to the Pontryagin densities

$$\theta(x)/(g^2 \Lambda) (k_A \text{tr} F_A^2 + k_G \text{tr} F_G^2 - k_L \text{tr} R^2) \quad (82)$$

that allows to identify the dual degree of freedom with the axion partner of the dilaton. The anomaly cancellation conditions are of course the same as before.

An interesting phenomenological consequence of this identification is the appearance of a Fayet-Iliopoulos term corresponding to the anomalous gauge group [22]. The $U(1)_A$ gauge transformation of the axion forces us to include the corresponding vector supermultiplet in the Kähler potential of the dilaton superfield to maintain its gauge invariance

$$\ln(S + S^\dagger) \rightarrow \ln(S + S^\dagger + \frac{c_A \Lambda}{48\pi^2 \Lambda'}) \quad (83)$$

This term contributes to the Euler-Lagrange equation of the auxiliary field D in the $U(1)_A$ vector multiplet

$$D_A = \frac{c_A}{48\pi^2 \kappa^2} + \sum_f q_f |\chi_f|^2 = 0 \quad (84)$$

where χ_f are the scalar partners of chiral fermions.

In order to avoid supersymmetry breaking by the anomalous D term at the unification scale, some $U(1)_A$ charged scalars must develop a v.e.v. that breaks the anomalous symmetry. If they correspond to flat directions of the superpotential so that supersymmetry is preserved, this mechanism provides interesting additional symmetry breaking that can reduce some of the large gauge groups resulting from compactification. It has also been used to explain the hierarchy of effective Yukawa couplings [24] [26] [25].

The presence of the constant in the effective D term has been proved [23] by determining a nonzero string 1-loop amplitude that can be identified with it. The coefficient found however, is proportional to $\sum_f n_f q_A^f$ instead of c_A . This seems to hint again that our cancellation condition (60) is right.

8 Summary

In summary, we have analysed in this paper the topological equivalence of different solutions of the Zumino-Stora descent equations, that are computed from a Chern-Simons 5-form defined up to an exterior derivative. Such solutions correspond to the same winding number of the Weyl determinant. Defining the anomaly as the gauge variation of the fermionic measure in the path integral, only one of these solutions can be properly called anomaly density. The physical equivalence of these functionals when used as anomaly densities has been tested, by studying the conditions on the spectrum of anomalous charges that we obtain when we impose the 4-dimensional Green-Schwarz mechanism. As expected, the conditions are the same. One of such conditions is not the usual one found in the literature, because we have been careful to include in the effective action all the Green-Schwarz terms that make the quantum theory truly gauge and Lorentz invariant.

Acknowledgments

We thank Prof. R. Shrock for encouragement and useful suggestions. We also thank Prof. M. Roček for enlightening discussions. This research was partially supported by the NSF grant PHY-93-09888.

References

- [1] K. Fujikawa, *Phys. Rev.* **D21** (1980)2848; *Phys. Rev.* **D29** (1984) 285; *Phys. Rev.* **D29** (1984) 285.
- [2] J. Wess and B. Zumino, *Phys. Lett.* **B37** (1971) 95.
- [3] M. B. Green and J. H. Schwarz, *Phys. Lett.* **B149** (1984) 117; M. B. Green, J. H. Schwarz and P. C. West, *Nuc. Phys.* **B254** (1985) 327.
- [4] B. Zumino, Y. Wu and A. Zee, *Nuc. Phys.* **B239** (1984) 477; B. Zumino *Lectures at Les Houches Summer School* (1983); R. Stora, *Lectures at Cargèse Summer School* (1983); L. Baulieu, *Nuc. Phys.* **B241** (1984) 557.
- [5] W. A. Bardeen, *Phys. Rev.* **184** (1969) 1848.
- [6] L. Baulieu, *Phys. Lett.* **B167** (1985) 56.
- [7] J. Preskill, *Ann. Phys.* **210** (1991) 323.
- [8] W. Lerche, B. E. W. Nilsson and A. N. Schellekens, *Nuc. Phys.* **B289** (1987) 609.
- [9] K. Itabashi, *Z. Phys* **C28** (1985) 601.
- [10] J. L. Petersen, *Act. Phys. Pol.* **B16** (1985) 271.
- [11] L. Alvarez-Gaumé and P. Ginsparg, *Nuc. Phys.* **B243** (1984) 449.
- [12] L. Alvarez-Gaumé and P. Ginsparg, *Ann. Phys.* **161** (1985) 423.
- [13] L. Alvarez-Gaumé, *An Introduction to anomalies in Fundamental problems of gauge field theory* Plenum Press N.Y. (1986) Ed.s G. Velo and A. S. Wightman
- [14] M. B. Einhorn and D. R. T. Jones, *Phys. Rev.* **D29** (1984) 331; S.-K. Hu, B. L. Young and D. W. McKay, *Phys. Rev* **D30** (1984)836.
- [15] E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, *Nuc. Phys.* **B195** (1982) 97; G. F. Chapline and N. S. Manton, *Phys. Lett.* **B120** (1983) 105.
- [16] P. Ginsparg, *Phys. Lett* **B197** (1987) 139.
- [17] J. A. Harvey and S. G. Naculich, *Phys. Lett.* **B217** (1989) 231.
- [18] J. Casas, E. K. Katehou and C. Muñoz, *Nuc. Phys.* **B317** (1989) 171.
- [19] H. Kawai, D. C. Lewellen, and S.-H.H. Tye, *Nuc. Phys.* **B288** (1987) 1.
- [20] J. L. Lopez and D. V. Nanopoulos, *Phys. Lett.* **B245** (1990) 111.
- [21] L. Ibañez, *Phys. Lett.* **B303** (1993) 55 (hep-ph/9205234).
- [22] M. Dine, N. Seiberg and E. Witten, *Nucl. Phys.***B289** (1987) 589.

- [23] J. Atick, L. Dixon, and A. Sen, *Nucl. Phys.***B292** (1987) 109; M. Dine, I. Ichinose, and N. Seiberg, *Nucl. Phys.* **B293** (1987) 253.
- [24] L. Ibañez and G. G. Ross, *Phys. Lett.* **B332** (1994) 100 (hep-ph/9403338).
- [25] P. Binétruy and P. Ramond, */em Phys. Lett.* **B350** (1995) 49 (hep-ph/9412385).
- [26] V. Jain and R. Shrock, *Phys. Lett.* **B352** (1995) 83 (hep-ph/9412367).